Dynamics and transport in the vicinity of a quantum phase transition

Félix Rose

PhD Advisor: Nicolas Dupuis

Laboratoire de physique théorique de la matière condensée,
Université Pierre et Marie Curie

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Introduction and outline

Some physical systems display a zero-temperature quantum phase transition when an external parameter is tuned
→ interesting playground for condensed matter theory: exotic scaling laws, models without quasiparticles...

Our goal: study the universal properties of the quantum $O(N)$ model in the vicinity of its quantum critical point.

Outline:

- Introduction: QPTs and the quantum $O(N)$ model.
- Motivation for a (nonperturbative) functional renormalization group approach.
- Thermodynamics of the model.
- The "Higgs" amplitude mode.
- Universal scaling function of the conductivity.

Quantum phase transitions — examples

Transverse field Ising model: $\hat{H} = -J_z \sum_{\langle ij \rangle} \hat{s}_i^z \hat{s}_j^z - h \sum_i \hat{s}_i^x$

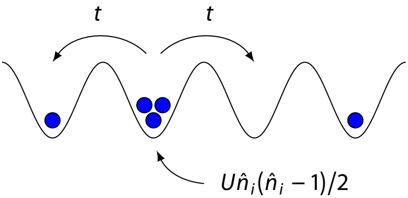
$(J_z/h)_c$ — J_z/h

Single paramagnetic GS:	Degenerate ferromagnetic GSs:
All spins aligned with **h** field	All spins parralel to z axis

Interacting lattice bosons (integer filling):

t: hopping

U: on-site interaction energy



$(t/U)_c$ — t/U

Insulating phase	Superfluid phase
(localized bosons)	(condensate)

Describing QPTs

We restrict ourselves to second order QPTs for which at the transition:

- the gap Δ closes;
- the correlation length ξ diverges.

We are interested in the low-energy, long distance physics → (effective) field theories.

The path integral formalism allows to rewrite the partition function.

Second quantization

$\hat{H}, \hat{\psi}(\mathbf{r})^\dagger, \hat{\psi}(\mathbf{r})$ operators

$$Z = \mathrm{Tr}\, e^{-\beta\hat{H}}$$

Path integral formulation

$\psi(\mathbf{r}, \tau)$ complex fields

$$Z = \int D[\psi^*, \psi]\, e^{-S[\psi^*, \psi]}$$

Statistical field theory and QPT

$$\hat{H}[\hat{\psi}^\dagger, \hat{\psi}] \to S[\psi^*, \psi] = \int_0^\beta \mathrm{d}\tau \left\{ H[\psi^*, \psi] + \int \mathrm{d}^d\mathbf{r}\, \psi^* \partial_\tau \psi \right\}.$$

Periodic BCs: $\psi(\mathbf{r}, \tau + \beta) = \psi(\mathbf{r}, \tau)$.

- Transforms the d-dimensional quantum problem into a $d + 1$ classical field theory...
- ... at the cost of a new imaginary time dimension $\tau \in [0, \beta]$.
- A QPT manifests as a phase transition in the classical field theory.

→ QPTs show all features of classical critical phenomenons: universality classes, scaling,...
E.g. for a $2d$ theory pressure and conductivity scale like

$$P(T) = P(T = 0) + T^{2+z} F\left(\frac{T}{\Delta}\right), \qquad \sigma(\omega) = \frac{q^2}{h} \Sigma\left(\frac{\omega}{\Delta}, \frac{T}{\Delta}\right)$$

with $F(x), \Sigma(x, y)$ universal scaling functions.

Quantum $O(N)$ model

Lorentz-invariant action, where $\boldsymbol{\varphi}$ is a real N-component field ($\sim \varphi^4$ model)

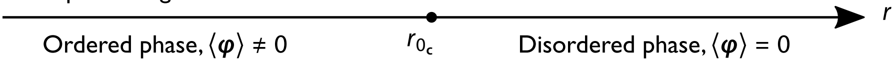
$$S[\boldsymbol{\varphi}] = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\mathbf{r} \left\{ \frac{1}{2} (\nabla\boldsymbol{\varphi})^2 + \frac{1}{2c^2} (\partial_\tau \boldsymbol{\varphi})^2 + r_0 \boldsymbol{\varphi}^2 + u_0 (\boldsymbol{\varphi}^2)^2 \right\}$$

- The couplings are temperature-independent
- Effective action that describes several condensed matter phase transitions:
  Bose–Hubbard model ($N = 2$), Quantum antiferromagnets ($N = 3$).

At zero T the model is strictly equivalent to the $d + 1$ classical $O(N)$ model.
Below a critical value of r_0, $\langle \boldsymbol{\varphi} \rangle \neq 0$

$\rightarrow$ the $O(N)$ symmetry is spontaneously broken.

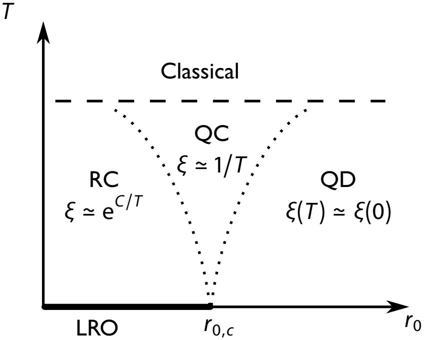
$T = 0$ phase diagram:



Qualitative $T > 0$ phase diagram

At finite T the QPT is destroyed (critical energy scales $\ll T$).

However there are crossovers between different regimes:



Typical ($N \geq 3$) $2d$ phase diagram.

- Energy scales: T and zero-T gap Δ.
- Crossover lines $T \sim \Delta \sim |\delta r_0|^\nu$.
- For $N = 1$ or 2 existence of a $T > 0$ "ordered" phase.

In the quantum critical fan, physical quantities are described by "exotic" scaling laws. E.g., in $2d$

$$P(T) = P(0) + T^{2+z} F_0, \qquad \sigma(\omega) = \frac{q^2}{h} \Sigma\left(\frac{\hbar\omega}{k_B T}\right).$$

Why do we need to go beyond mean-field?

Like for classical phase transitions, the dimension plays an important role.

- For $d + 1 \geqslant 4$, the theory is controlled by a Gaussian fixed point and the MF is qualitatively correct.
- For $d = 2$ it is harder to answer. Strongly interacting Wilson–Fischer fixed point: MF is wrong.

Perturbation theories (weak/strong coupling, large-N) ill-suited to study the critical regime.

→ need for a nonperturbative framework: FRG

Complementary existing approaches:

- Analytical: holographic models, conformal field theories.
- Numerical: Monte Carlo, exact diagonalization.

The functional renormalization group: philosophy

FRG: implemented on effective action (Gibbs free energy) $\Gamma[\boldsymbol{\varphi}]$. $(\equiv \Gamma[\langle\boldsymbol{\varphi}\rangle])$.

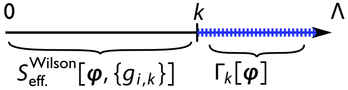
Γ: Legendre transform of free energy $\ln Z$.

$$Z[\mathbf{J}] = \int D[\boldsymbol{\varphi}] \exp\left(-S[\boldsymbol{\varphi}] + \int_x \mathbf{J}\cdot\boldsymbol{\varphi}\right), \qquad \Gamma[\boldsymbol{\varphi}] = -\ln Z[\mathbf{J}] + \int_x \mathbf{J}\cdot\boldsymbol{\varphi}.$$

Vertices $\Gamma^{(n)} \equiv \delta^n\Gamma/\delta\varphi^n$ contain the physical information:

- $\Gamma_{\boldsymbol{\varphi}(\mathbf{x})=\text{const.}} = U$: potential $\to$ thermodynamics (physical $\boldsymbol{\varphi}$ extremizes Γ).
- $\Gamma^{(2)} = [G]^{-1}$: inverse propagator.

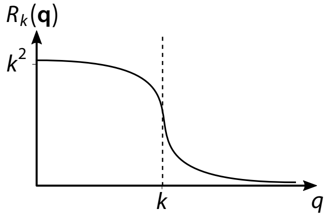
Similar in concept to Wilsonian RG: degrees of freedom are progressively integrated out.



The functional renormalization group: in practice

This is implemented by adding to the action a "mass-like" term:

$$S \to S_k = S + \Delta S_k,$$
$$\Delta S_k[\boldsymbol{\varphi}] = \frac{1}{2}\int_{\mathbf{q}} \boldsymbol{\varphi}(\mathbf{q}) \cdot R_k(\mathbf{q})\boldsymbol{\varphi}(\mathbf{q}).$$



R_k gives a very large mass to modes at momenta $\lesssim k$.

New k-dependent effective action $\Gamma \to \Gamma_k$.

- $k = \Lambda$: all fluctuations are frozen and MF is exact, $\Gamma_\Lambda = S$.
- $k = 0$: all fluctuations are taken into account, $\Gamma_{k=0} = \Gamma$.

Exact flow equation

$$k\partial_k \Gamma_k[\boldsymbol{\varphi}] = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k \left(\Gamma_k^{(2)}[\boldsymbol{\varphi}] + R_k\right)^{-1}\right\}$$

Functional renormalization group: approximation procedures

Simplest idea: the derivative expansion (DE).

$$\textit{Ansatz:}\ \Gamma_k[\boldsymbol{\varphi}] = \int_{\mathbf{x}} \frac{Z_k(\boldsymbol{\varphi}^2)}{2}(\partial_\mu \boldsymbol{\varphi})^2 + U_k(\boldsymbol{\varphi}^2) + \frac{Y_k(\boldsymbol{\varphi}^2)}{4}(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})^2.$$

- Nonperturbative (no small parameter) and 1-loop exact.
- Coupled PDEs for U_k, Z_k and Y_k: numerically solvable.
- Valid in principle at small momenta only.

Other example: the Blaizot–Mendéz-Galain–Wschebor procedure (BMW) where high-order vertices $\Gamma^{(3)}, \Gamma^{(4)}, \ldots$ are approximated → closed equation for $\Gamma^{(2)}(\mathbf{p})$.

Yields good results at finite momenta! **[Blaizot *et al.*, PRE '06] [Benitez *et al.*, PRE '09]**

Critical exponents for the $(2+1)d$ Ising model:

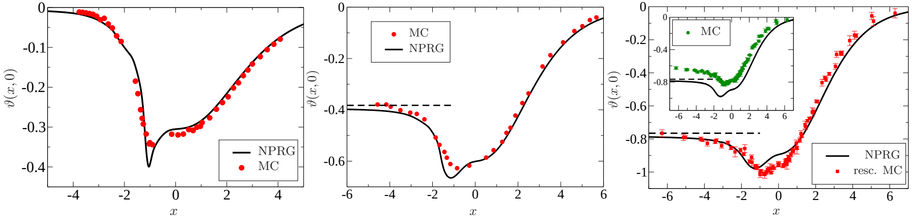
	FRG DE	FRG BMW	Bootstrap
η	.0443	.039	.036298(2)
ν	.6307	.632	.629971(4)

Thermodynamics

Application of the DE: we compute the universal scaling functions of the pressure and internal energy density.

$$P(T) = P(T=0) + \frac{(k_B T)^3}{(\hbar c)^2} F(\Delta/(k_B T), \qquad \epsilon(T) = \epsilon(T=0) - \frac{(k_B T)^3}{(\hbar c)^2}\theta(\Delta/k_B T).$$

Left to right: $\theta(x)$ for Ising, XY, Heisenberg $[x = (\Delta/k_B T)^{1/\nu}]$.



[Rançon, F. R. et al., PRB '16].

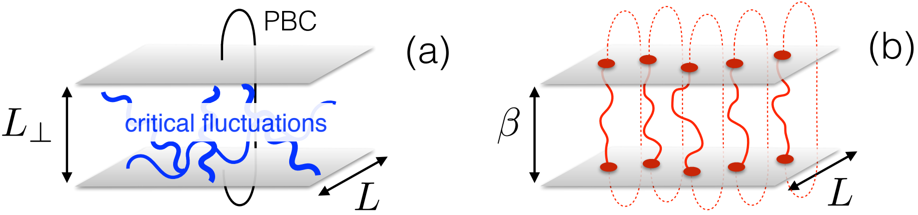
Full lines: FRG; dots: Monte-Carlo simulations for 3D classical spin systems with PBC.

Full potential $O(\partial^2)$ DE, improvement over previous work by Rançon et al.

Classical — quantum mapping

A classical theory confined along a $L_\perp$ direction and a quantum $T > 0$ theory are equivalent!

$$H_{\text{cl.}} = \int \mathrm{d}^{D-1}\mathbf{x}_{||} \int_0^{L_\perp} \mathrm{d}x_\perp \, H, \qquad H_{\text{q.}} = \int \mathrm{d}^d\mathbf{r} \int_0^{\hbar\beta} \mathrm{d}\tau \, H.$$



The scaling function θ describes the scaling of the critical Casimir force of a $3D$ classical model near criticality with periodic boundary conditions.

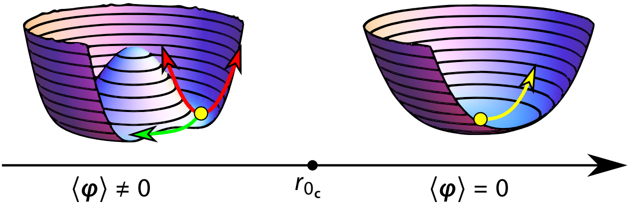
$$\text{Casimir force } f(L_\perp, \xi) \sim L_\perp^{-D} \theta(L_\perp/\xi).$$

(Critical Casimir forces: **[Fisher and de Gennes, C.R. Acad. Sci. '78]**) ξ: correlation length.

Dynamics: mean-field excitations

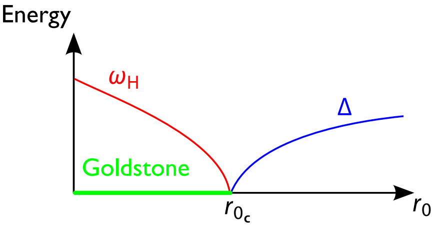
$$U(\boldsymbol{\varphi}) = r_0 \boldsymbol{\varphi}^2 + u_0 (\boldsymbol{\varphi}^2)^2$$

Mean-field zero T phase diagram:



$N-1$ Goldstone modes
+ amplitude mode (gap ω_H).

N gapped modes (gap Δ).



Mean-field:

$$\Delta = A \left| r_0 - r_{0_c} \right|^{1/2}$$

$$\omega_H = A\sqrt{2} \left| r_0 - r_{0_c} \right|^{1/2}$$

Beyond the Gaussian approximation

Question: what happens to the "Higgs" amplitude mode beyond MF?

- Is it a well defined mode? What is the quasiexcitation lifetime?
- What happens near the critical point?

> "In general, this Higgs particle can decay into multiple lower-energy spin waves. It has been argued that such decay processes dominate for $d < 3$, and the Higgs particle is therefore not a stable excitation."

[Sachdev, *Quantum Phase Transitions*, 2nd. ed]

Emission of Goldstone bosons → IR divergence of the longitudinal susceptibility.

[Patasinskij *et al.*, JETP' 73], **[Zwerger, PRL '04]**, **[Dupuis, PRE '11]**, ...

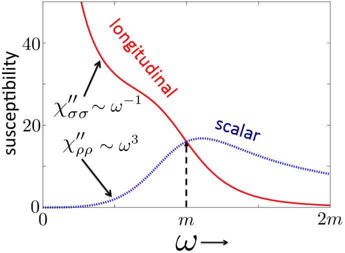
Scalar response function

Answer: consider a different response function. **[Podolsky, Auerbach and Arovas, PRB ’11]**

Right probe: the scalar susceptibility.

$$\chi_s(\mathbf{r},\tau) = \left\langle \boldsymbol{\varphi}^2(\mathbf{r},\tau)\boldsymbol{\varphi}^2(0,0)\right\rangle,$$

$$\chi''_s(\omega) = \mathrm{Im}[\chi_s(\mathbf{q}=0, i\omega_n \to \omega + i0^+)].$$



To compute the 4-point correlation function: introduce an additional source,

$$S \to S[\mathbf{J}, h] = S + \int d^{d+1}x\, \mathbf{J}\cdot\boldsymbol{\varphi} + \int d^{d+1}x\, h\boldsymbol{\varphi}^2.$$

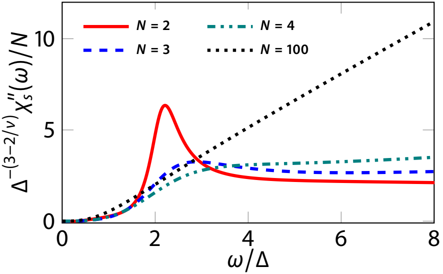
Effective action $\Gamma[\boldsymbol{\varphi}, h]$: Legendre transform wrt $\mathbf{J}$ but not h. Then

$$\chi_s(\omega) = -\Gamma^{(0,2)}(\omega) + \left(\Gamma^{(2,0)}(\omega)\right)^{-1}\left(\Gamma^{(1,1)}(\omega)\right)^2, \qquad \Gamma^{(n,m)} = \frac{\delta^{(n+m)}\Gamma}{\delta^{(n)}\varphi\,\delta^{(m)}h}.$$

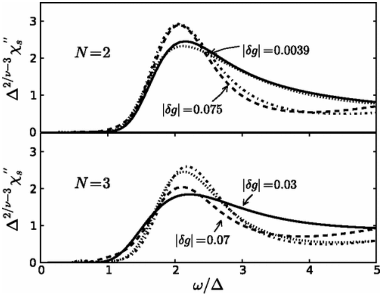
Approximation method: BMW to obtain full frequency dependence.

Results and comparison

FPRG BMW **[Rose and Dupuis, PRB '15]**:



MC **[Gazit, Podolski, Auerbach PRL '13]**:



Experimental observation: **[Endres et al., Nature '12]**.

Higgs mass m_H/Δ	$N = 3$	$N = 2$		$N = 3$	$N = 2$
MF	$\sqrt{2}$	$\sqrt{2}$	FRG BMW	2.7	2.2
QMC **(Chen et al.)**		3.3(8)	Lattice QMC **(Löhofer et al.)**	2.6(4)	
QMC **(Gazit et al.)**	2.2(3)	2.1(3)	Exact diag. **(Nishiyama)**	2.7	2.1(2)
ϵ expansion **(Katan et al.)**	1.64	1.67			

Conductivity of the O(N) model

O(N) symmetry → conservation of angular momentum L, current $\partial_t L + \nabla \cdot \mathbf{J} = 0$.

We make the $O(N)$ symmetry local by adding a gauge field, $\partial_\mu \to D_\mu = \partial_\mu - A_\mu$.

$$A_\mu = A_\mu^a T^a \in \mathrm{so}(N) \qquad T^a\text{: } N(N-1)/2 \text{ generators, } T_{ij}^a = -T_{ji}^a$$

Current densities $J_\mu^a = -\dfrac{\delta S}{\delta A_\mu^a} = j_\mu^a - A_\mu^a \boldsymbol{\varphi} \cdot T^a \boldsymbol{\varphi}$, $\qquad j_\mu^a = \boldsymbol{\varphi} \cdot T^a \partial_\mu \boldsymbol{\varphi}$

$N = 2$ (bosons): $\mathbf{j} \sim \mathrm{i}(\psi^* \nabla \psi - \psi \nabla \psi^*)$, $\quad \psi = \varphi_1 + \mathrm{i}\varphi_2$.

Linear response theory

$$K_{\mu\nu}^{ab}(\mathbf{x}-\mathbf{x}') = \left\langle j_\mu^a(\mathbf{x}) j_\nu^b(\mathbf{x}') \right\rangle - \delta_{\mu\nu}\delta(\mathbf{x}-\mathbf{x}') \left\langle T^a \boldsymbol{\varphi} \cdot T^b \boldsymbol{\varphi} \right\rangle = \frac{\delta^{(2)} \ln Z}{\delta A_\mu^a(\mathbf{x}) \delta A_\nu^b(\mathbf{x}')}$$

$$\sigma_{\mu\nu}^{ab}(\mathrm{i}\omega_n) = -\frac{1}{\omega_n} K_{\mu\nu}^{ab}(p_x = 0, p_y = 0, p_z = \mathrm{i}\omega_n)$$ conductivity tensor

Conductivity: generalities

The conductivity tensor $\sigma_{\mu\nu}^{ab}$:

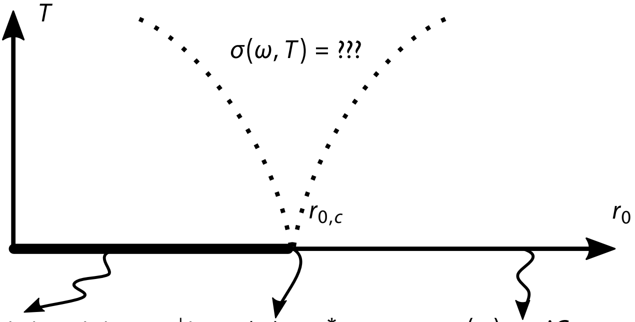
- is diagonal, $\sigma_{\mu\nu}^{ab} = \delta_{\mu\nu}\delta_{ab}\sigma^{aa}$;
- has two independent components, $\sigma^{aa}(\omega) = \begin{cases} \sigma_A(\omega) & \text{if } T^a\boldsymbol{\varphi} \neq 0, \\ \sigma_B(\omega) & \text{if } T^a\boldsymbol{\varphi} = 0; \end{cases}$
- in the disordered phase and at the QCP $\sigma_A = \sigma_B = \sigma$.

For $N = 2$, there is only one $\mathrm{so}(N)$ generator and the conductivity in the ordered phase reduces to σ_A.

Universal properties

Long-term objective: determine the conductivity in the QC regime.

Low frequency behavior:



$\sigma_A(\omega) \simeq i/L(\omega + i0^+)$ $\quad \sigma(\omega) = \sigma^*$ $\quad \sigma(\omega) \simeq -iC\omega$

$\sigma_B(\omega) = ?$ (not much investigated as only exists for $N \geq 3$)

$\sigma^*/\sigma_q, C/L\sigma_q^2$ are universal! $\quad (\sigma_q = q^2/h)$ **[Fischer *et al.*, PRL '89]**

Goals

Objective: determine the universal scaling form of the conductivity.
Technically: compute 4 point correlation functions $\left\langle j_\mu^a j_\nu^b \right\rangle$.

Approaches:

- QMC **(Sørensen, Chen, Prokof'ev, Pollet, Gazit, Podolsky, Auerbach)**;
- Holography **(Myers, Sachdev, Witzack-Krempa)**;
- CFT **(Poland, Sachdev, Simmons-Duffin, Witzack-Krempa)**;
- FRG (us!).

Effective action formalism

Idea: use the same trick than for the Higgs.

The partition function depends on two sources, the gauge field and **J** that couples linearly to $\boldsymbol{\varphi}$:

$$Z[\mathbf{J},\mathbf{A}] = \int D[\boldsymbol{\varphi}] \exp(-S[\boldsymbol{\varphi},\mathbf{A}] + \int_{\mathbf{x}} \mathbf{J}\cdot\boldsymbol{\varphi}).$$

The effective action is the Legendre transform of $\ln Z$ wrt **J** *but not* **A**:

$$\Gamma[\boldsymbol{\varphi},\mathbf{A}] = -\ln Z[\mathbf{J},\mathbf{A}] + \int_{\mathbf{x}} \mathbf{J}\cdot\boldsymbol{\varphi}.$$

$$K^{ab}_{\mu\nu} = \frac{\delta^{(2)}\ln Z}{\delta A^a_\mu \delta A^b_\nu} = -\Gamma^{(0,2)}_{a\mu,b\nu} + \Gamma^{(1,1)}_{i,a\mu}\left(\Gamma^{(2,0)}\right)^{-1}_{ij}\Gamma^{(1,1)}_{j,b\nu}$$

with

$$\Gamma^{(n,m)} = \frac{\delta^{(n+m)}\Gamma}{\delta^{(n)}\varphi\,\delta^{(m)}A}.$$

ERG formulation

Problem of the FRG: the regulator (~ mass) breaks down gauge invariance:

$$\Delta S_k = \frac{1}{2}\int_{\mathbf{q}} \boldsymbol{\varphi}(\mathbf{q}) \cdot R_k(\mathbf{q}^2)\boldsymbol{\varphi}(-\mathbf{q}).$$

How to recover gauge invariance?

Answer: make the regulator gauge dependent!

[Morris, N. Phys. B '00] **[Codello, Percacci *et al.*, EPJC '16]** **[Bartosh, PRB '13]**

$$\Delta S_k = \frac{1}{2}\int_{\mathbf{x}} \boldsymbol{\varphi}(\mathbf{x}) \cdot R_k(-\partial_\mu^2)\boldsymbol{\varphi}(\mathbf{x}) \to \Delta S_k[\mathbf{A}] = \frac{1}{2}\int_{\mathbf{x}} \boldsymbol{\varphi}(\mathbf{x}) \cdot R_k(-D_\mu^2)\boldsymbol{\varphi}(\mathbf{x})$$

Modified flow equations due to the presence of **A**.

RG approximation scheme

Which approximation procedure do we use?

First idea: BMW to obtain full momentum dependence (as done for the study of the Higgs mode).

Problem: it fails!

- Impossible to close the flow equations rigorously.
- Setting momenta to zero in flow equations breaks down gauge invariance.

Derivative expansion scheme

We try a derivative expansion scheme and project the flow equation onto a gauge-invariant *Ansatz*. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$ allows us to build two $O(A_\mu^2)$ gauge-invariant terms:

$$\Gamma_k[\boldsymbol{\varphi}, \mathbf{A}] = \int_{\mathbf{x}} \frac{1}{2} Z_k(\rho)(D_\mu \boldsymbol{\varphi})^2 + \frac{1}{4} Y_k(\rho)(\partial_\mu \rho)^2 + U_k(\rho) \quad \text{(standard } O(\partial_\mu^2) \text{ DE)}$$
$$+ \frac{1}{4} X_{1,k}(\rho)\,\mathrm{Tr}(F_{\mu\nu}^2) + \frac{1}{4} X_{2,k}(\rho)(F_{\mu\nu}\boldsymbol{\varphi})^2.$$

$\rho = \boldsymbol{\varphi}^2/2$, $\rho_{0,k}$: minimum of the potential.

Expression of the conductivity within the DE

$$\sigma_A(\omega) = 2\rho_0 Z(\rho_0)/(\omega + i0^+) + \omega[X_1(\rho_0) + 2\rho_0 X_2(\rho_0)],$$
$$\sigma_B(\omega) = \omega X_1(\rho_0).$$

Results

This simple DE scheme allows us to recover the low momenta physics! We retrieve the universal ratio C/L. Exact value for $N = \infty$, good agreement with MC (~ 5%) for $N = 2$.

N	2	3	4	1000	∞ (exact)
$C/NL\sigma_q^2 \quad (\sigma_q = q^2/h)$	0.105	0.0742	0.0598	0.0416	0.04167

The picture is more complicated in the critical regime.

- DE can sometimes be extrapolated to finite ω but not always.
- $\Gamma^{(0,2)}(\omega) \sim 1/\omega$: divergence in the flow:

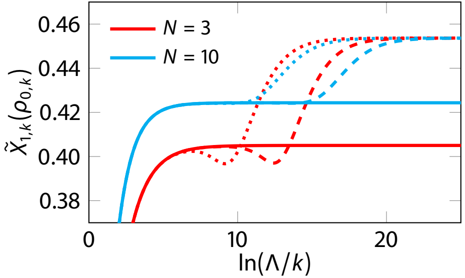
$$\sigma(\omega) \sim \tilde{X}^*_{1,\text{crit}} \frac{\omega}{k} \qquad \text{with } \tilde{X}^*_{1,\text{crit}} = \lim_{k\to 0} k X_1(\rho_{0,k}) \quad \text{(fixed point value)}.$$

- Setting $\omega \sim k$ yields an estimate of the conductivity, $\sigma^* \sim \tilde{X}^*_{1,\text{crit}}$.

σ_B

Similarly, in the ordered phase, $\sigma_B(\omega) \sim \tilde{X}^*_{1,\mathrm{ord}}\omega/k$.

$\tilde{X}^*_{1,\mathrm{ord}}$ is an universal number — verified for $N = \infty$, conjecture for $N < \infty$.



Full: QCP

Dashed: ordered

More surprising: $\tilde{X}^*_{1,\mathrm{ord}}$ numerically does not depend on N!

$$\sigma_B(\omega) = \frac{\pi}{8}\sigma_q \text{ for all } N\text{: “superuniversality”!}$$

[Rose and Dupuis, PRB ’17]

Beyond DE...?

Next project: use a momentum-dependent scheme to compute $\sigma(\omega)$ at finite ω.

From this: deduce $\sigma_B, \sigma^*, ...$

Idea: DE-like *Ansatz* with momentum dependence.

Preliminary results:

- $\sigma^* \simeq 0.34\sigma_q$, bootstrap result: $\sigma^* = 0.3554(6)\sigma_q$ **[Kos et al., JHEP '15]**, QMC predicts value from 0.355 to 0.361.
- The conjecture that $\sigma_B(\omega \to 0) = \frac{\pi}{8}\sigma_q$ for all N holds.

Review and conclusion

- Although harder than thermodynamics FRG can be successfully used to describe dynamics in the vicinity of QPTs and compare well with complementary approaches (holography, CFTs, numerics).
- Study of the excitation spectrum of the O(N) model: Higgs **[Rose et. *al*, PRB '15]** (and bound states **[Rose et. *al*, PRB '16]**).
- Transport: a simple derivative expansion allows to obtain results that compare well with MC. Results allow to make a conjecture on the universal behavior of σ_B **[Rose and Dupuis, PRB '17]**.
- This will soon be confirmed with a momentum-dependent scheme we are developing!
- Hardest part (and long term goal): $T > 0$, a little because of computational time and a lot because of analytic continuation! Proposals to overcome this difficulty **(Strodthoff, Pawlowski)**.